https://www.linkedin.com/feed/update/urn:li:activity:6560017009399205888

(**a**) Let a,b,c be the positive real numbers such that $a^2+b^2+c^2+2abc=1$. Show that there exists an acute triangle ABC such that

$a=\cos A, b=\cos B, c=\cos C$.

(**b**) Let $a,b,c\geq 0$ with $a^2+b^2+c^2+2abc=1$. Show that there are $A,B,C\in[0,\pi/2]$ with $a=\cos A, b=\cos B, c=\cos C$, and $A+B+C=\pi$.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

First note that if $a,b,c\geq 0$ (case (**b**)) then $a^2+b^2+c^2+2abc=1$ implies $a,b,c\leq 1$ and, therefore, angles $\alpha:=\arccos a,\ \beta:=\arccos b, \gamma:=\arccos c$ belongs to $[0,\pi/2]$.

If $a,b,c>0$ (case (**a**)) then $a^2+b^2+c^2<1$ implies $a,b,c<1$ and we have $a,b,c\in(0,1)\iff\alpha,\beta,\gamma\in(0,\pi/2)$.

In both cases equation $a^2+b^2+c^2+2abc=1$ becomes equivalent to equation

(**1**) $\quad\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1$,

where $\alpha,\beta,\gamma\in(0,\pi/2)$ in the case (**a**) and $\alpha,\beta,\gamma\in[0,\pi/2]$ in the case case (**b**).

We have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma-1=$

$\cos^2\gamma+\dfrac{2\cos^2\alpha-1+2\cos^2\beta-1}{2}+\cos\gamma(\cos(\alpha+\beta)+\cos(\alpha-\beta))=$

$\cos^2\gamma+\dfrac{\cos 2\alpha+\cos 2\beta}{2}+\cos\gamma(\cos(\alpha+\beta)+\cos(\alpha-\beta))=$

$\cos^2\gamma+\cos(\alpha+\beta)\cos(\alpha-\beta)+\cos\gamma(\cos(\alpha+\beta)+\cos(\alpha-\beta))=$

$(\cos\gamma+\cos(\alpha+\beta))(\cos\gamma+\cos(\alpha-\beta))$.

Since $\cos\gamma+\cos(\alpha+\beta)=2\cos\dfrac{\alpha+\beta+\gamma}{2}\cos\dfrac{\alpha+\beta-\gamma}{2}$ and $\cos\gamma+\cos(\alpha-\beta)=$ $2\cos\dfrac{\gamma+\alpha-\beta}{2}\cos\dfrac{-\alpha+\beta+\gamma}{2}$ then, denoting $\varphi:=\dfrac{\alpha+\beta+\gamma}{2}$ we obtain

(**1**)$\iff 4\cos\varphi\cdot\cos(\varphi-\gamma)\cdot\cos(\varphi-\beta)\cdot\cos(\varphi-\alpha)=0$.

In the case (**a**) since $\cos(\varphi-\gamma),\cos(\varphi-\beta),\cos(\varphi-\alpha)\neq 0$ (because $\alpha,\beta,\gamma\in(0,\pi/2)$ implies $\varphi-\gamma,\varphi-\beta,\ \varphi-\alpha\in(-\pi/2,\pi/2)$) we obtain that (**1**)$\iff\cos\varphi=0\iff$ $\varphi=\dfrac{\pi}{2}\iff\alpha+\beta+\gamma=\pi$.

Thus, in this case exists an acute triangle ABC such that $(A,B,C)=(\alpha,\beta,\gamma)$.

Consider now case (**b**).

Since in this case if, for example, $b=0$ and $c=0$ then $a=1$ and, therefore, $\alpha+\beta+\gamma=\pi$.

If only one of the numbers a,b,c equal zero, let it be $c=0$ then $\gamma=\pi/2$ and $a^2+b^2=1$ implies $a,b\in(0,1)\iff\alpha,\beta\in(0,\pi/2)$ and, therefore, $\varphi-\gamma,\varphi-\beta,\ \varphi-\alpha\in(-\pi/2,\pi/2),\ \varphi\in(0,3\pi/4)$.

Thus, (**1**)$\iff\cos\varphi=0\iff\alpha+\beta+\gamma=\pi$ again.